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Estimates of Logarithmic Sobolev Constant for Finite-Volume Continuous Spin Systems

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Let M be a compact, connected Riemannian manifold (with or without boundary); we study the logarithmic Sobolev constant for stochastic Ising models on $M^{\mathbf{Z}^d}$. Let $\{\Lambda\}$ be a sequence of cubes in $\mathbf{Z}^d$; we show that the logarithmic Sobolev constant for the finite systems on M^Λ shrinks at most exponentially fast in $|\Lambda|^{(d-1)/d}$ $(d \geqslant 2)$, which is sharp in order for the classical Ising models with $M = [-1, 1]$. Moreover, a geometrical lemma proved by L. E. Thomas is also improved.

KEY WORDS: Logarithmic Sobolev constant; spectral gap; stochastic Ising model; diffusion process; Peierls' contour; Gibbs state.

1. INTRODUCTION

In recent years, a number of papers have studied the logarithmic Sobolev constant and the spectral gap for lattice spin systems (see refs. 11 and 12 and references therein). It is shown in ref. 12 that the logarithmic Sobolev inequality fails for an infinite lattice system whenever the phase transitions occur. So, in the phase coexistence region, what we are mainly interested in is the asymptotic behavior of these two constants for finite-volume spin systems as the volume goes to the full lattice space. This topic was studied by several authors for the classical Ising models with single spin space $\{-1, 1\}$. For example, ref. 18 proved that the logarithmic Sobolev inequality always holds for one-dimensioal systems, and the upper bound and the lower bound of the spectral gap for finite-volume systems were obtained, respectively, in refs. 13 and 10, which then were improved in ref. 9 for $d=2$ by using the theory of surface tension developed in ref. 3. Moreover, the logarithmic Sobolev constant for finite-volume systems with discrete spins was also studied in recent work.[17]

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In this paper, we study the logarithmic Sobolev constant and the spectral gap for finite-volume spin systems with a compact Riemannian manifold as single spin space. The resulting estimates are quite similar to those for the discrete spin case. Roughly speaking, let Λ be a cube in $\mathbf{Z}^d$; then the logarithmic Sobolev constant for spin systems with volume Λ shrinks at most exponentially fast in $|\Lambda|^{(d-1)/d}$ $(d \geqslant 2)$. Moreover, for the classical Ising model with single spin space $[-1, 1]$, the corresponding spectral gap does decay in this order when the temperature is low enough.

Let M be a compact, connected Riemannian manifold with or without boundary, and let $\mathscr{J} = \{J_\Lambda : \varnothing \neq \Lambda \subset \mathbf{Z}^d\}$ be a shift-invariant, smooth potential with range $R \geqslant 1$. That is, for each $\Lambda \subset \mathbf{Z}^d$, $J_\Lambda \in C^\infty(M^{\mathbf{Z}^d})$, which depends only on the coordinates in Λ, $J_{\Lambda+u} = J_\Lambda \circ \theta_{-u}$, $u \in \mathbf{Z}^d$, where θ denotes the natural shift on $\mathbf{Z}^d$, and $J_\Lambda \equiv 0$ if $d(\Lambda) := \sup_{u,v \in \Lambda} |u - v| > R$, where $|\cdot|$ is the Euclidean norm. Appoint $J_\varnothing \equiv 0$, for $u \in \mathbf{Z}^d$, let $H_u = \beta \sum_{\Lambda \ni u} J_\Lambda$ $(\beta > 0)$, and define L_∞ by

$$L_\infty f = \sum_{u \in \mathbf{Z}^d} (\Delta_u f - \langle \nabla_u H_u, \nabla_u f \rangle)$$

for a cylindrically smooth function f, where Δ_u and ∇_u refer, respectively, to the Laplace–Beltrami and gradient operators on the uth manifold. Then the set of all reversible measures for the L_∞-diffusion process (with reflecting boundary if $\partial M \neq \varnothing$) coincides with that of all Gibbs states with potential $\beta \mathscr{J}$.[2]

We say that a logarithmic Sobolev inequality holds with respect to a Gibbs state μ if there exists $\alpha > 0$ such that

$$\mu(f^2 \log f^2) \leqslant \frac{2}{\alpha} \mu\left(\sum_{u \in \mathbf{Z}^d} \|\nabla_u f\|^2 \right) + \mu(f^2) \log \mu(f^2)$$

holds for all cylindrically smooth f, where $\mu(f) = \int f \, d\mu$. The logarithmic Sobolev constant, denoted by $\alpha(L_\infty)$, is the maximum of α.

It is known that the Gibbs state uniquely exists and $\alpha(L_\infty) > 0$ provided $d = 1$ or β is sufficiently small.[2,7,11,12,15,16] So, we will be mainly restricted to the case that $d \geqslant 2$ and β is large enough.

Let ν be the volume element on M and let Λ be a cube in $\mathbf{Z}^d$ with side length $l \geqslant 1$. For $y \in M^{\Lambda^c}$, define

$$H_{\Lambda|y}(x) = \beta \sum_{F \subset \mathbf{Z}^d} J_F(x \times y), \qquad x \in M^\Lambda$$

Let

$$L_{\Lambda|y}=\sum_{u\in\Lambda}(\Delta_u-\nabla_u H_{\Lambda|y}),\qquad d\mu_{\Lambda|y}=Z_{\Lambda|y}^{-1}\exp(-H_{\Lambda|y})\,dv^{\Lambda}$$

where $Z_{\Lambda|y}=\int_{M^\Lambda}\exp(-H_{\Lambda|y})\,dv^{\Lambda}$. Then the $L_{\Lambda|y}$-diffusion process (with reflecting boundary if $\partial M\neq\varnothing$) is reversible with respect to the unique stationary measure $\mu_{\Lambda|y}$. Denote by $\alpha(L_{\Lambda|y})$ the logarithmic Sobolev constant with respect to $\mu_{\Lambda|y}$, which is described as the largest α such that

$$\mu_{\Lambda|y}(f^2\log f^2)\leqslant\frac{2}{\alpha}\mu_{\Lambda|y}\left(\sum_{u\in\Lambda}\|\nabla_u f\|^2\right)+\mu_{\Lambda|y}(f^2)\log\mu_{\Lambda|y}(f^2)\quad(1.1)$$

holds for all $f\in C^{\infty}(M^{\Lambda})$.

Next, for the free boundary case, let $\alpha(L_\Lambda)$ be defined as $\alpha(L_{\Lambda|y})$ upon replacing $H_{\Lambda|y}$ by $H_\Lambda=\beta\sum_{F\subset\Lambda}J_F$.

For $f\in C(M^\Lambda)$, let $\delta(f)=\max f-\min f$. Define $\lambda=\sum_{A\ni 0,|A|\geqslant 2}\delta(J_A)$ and let $\alpha_0(\beta)$ be the logarithmic Sobolev constant referring to $\Delta_0-\beta\nabla_0 J_{\{0\}}$. Since M is compact, $\alpha_0(\beta)>0$ for all β (see refs. 1, 2, and 14 for detailed estimates). One of the main results of this paper is the following.

Theorem 1.1. For $d\geqslant 2$ we have

$$\alpha(L_\Lambda)\geqslant\alpha_0(\beta)\exp[-2\beta\lambda Rdc_d(l+1)^{d-1}]$$
$$\alpha(L_{\Lambda|y})\geqslant\alpha_0(\beta)\exp[-2\beta\lambda Rd(1+c_d)(l+1)^{d-1}],\qquad y\in M^{\Lambda^c}$$

where $c_d=[1+(d-1)^{-1}][(d-1)^{1/(d-1)}+(d-1)^{-1}]\leqslant 4$.

It should be pointed out that the same type of estimation may be obtained by using the method of ref. 17, in which discrete spins were considered, but we prefer to go along a slightly different line, for which the resulting estimates (Theorem 1.1) are finer and the idea is also very natural. In contrast with the expected result by using the method of ref. 17, an obvious merit of Theorem 1.1 is that the lower bounds depend on the amplitude λ rather than the uniform norm of the gradient of potential.

As for the upper-bound estimation, we adopt the Peierls contour argument used in ref. 13. However, since the spin space is now continuous, more analysis techniques are necessary. On the other hand, to make the Peierls argument available in the present case, we restrict ourselves to a specific model for which the phase transitions occur at low temperatures.[4] Let $M=[-1,1]$ and

$$J_A(x)=\begin{cases}-\beta x_u x_v & \text{if}\quad A=\{u,v\}\text{ with }|u-v|=1\\ 0 & \text{otherwise}\end{cases}\qquad(1.2)$$

For given cube $\Lambda \subset \mathbf{Z}^d$ with side length l, let

$$H_\Lambda(x) = -\sum_{u, v \in \Lambda, |u-v|=1} \beta x_u x_v$$

$$L_\Lambda = \sum_{u \in \Lambda} \left(\frac{\partial^2}{\partial x_u^2} - \frac{\partial H_\Lambda}{\partial x_u} \cdot \frac{\partial}{\partial x_u} \right)$$

Define

$$Z_\Lambda = \int_{[-1,1]^\Lambda} \exp(-H_\Lambda(x))\, dx$$

$$\mu_\Lambda(dx) = Z_\Lambda^{-1} \exp(-H_\Lambda(x))\, dx$$

Then the spectral gap for the reflecting L_Λ-diffusion process is

$$\text{gap}(L_\Lambda) = \inf\{\mu_\Lambda(\|\nabla_\Lambda f\|^2): f \in C^1([-1, 1]^\Lambda) \text{ with } \mu_\Lambda(f) = 0 \text{ and } \mu_\Lambda(f^2) = 1\}$$

where $\|\nabla_\Lambda f\|^2 = \sum_{u \in \Lambda} (\partial f / \partial x_u)^2$.

Note that $\text{gap}(L_\Lambda) \geqslant \alpha(L_\Lambda)$; then the next result shows that the estimate given in Theorem 1.1 can be sharp in order.

Theorem 1.2. For model (1.2), there exist $\alpha > 0$ and $c(\beta) > 0$ such that

$$\text{gap}(L_\Lambda) \leqslant c(\beta) \exp[-\alpha \log(\beta / \log \beta)(l+1)^{d-1}]$$

holds for all $l > 1$ and sufficiently large β.

A referee has pointed out that the dependence on β of the above upper bound is different from that of the lower bound given in Theorem 1.1. So it is not sure yet whether this upper bound is also optimal in β for the present model.

2. PROOF OF THEOREM 1.1

The idea of the proof is in some sense a combination of the methods used in refs. 17 and 18. To complete the proof, we need some lemmas. The first three are similar to those given in ref. 17.

Lemma 2.1. Let M_1 and M_2 be two compact Riemannian manifolds; for $V_i \in C^2(M_i)$, let $L_i = \Delta_i + \nabla_i V_i$, where Δ_i and ∇_i refer to the corresponding operators on M_i $(i = 1, 2)$. Define

$$Lf(x, y) = L_1 f(\cdot, y)(x) + L_2 f(x, \cdot)(y), \qquad f \in C^2(M_1 \times M_2)$$

Then $\alpha(L) = \min\{\alpha(L_1), \alpha(L_2)\}$.

Next, note that for all $\mu \in \mathscr{P}(M)$ and positive $f \in C(M)$,

$$\int_M f \log \frac{f}{\mu(f)} \, d\mu = \inf\{\mu(f \log f - f \log t - f + t) \colon t > 0\}$$

and $f \log f - f \log t - f + t \geqslant 0$ for all $t > 0$. Then the next lemma follows from (1.1) (see ref. 2).

Lemma 2.2. For $U, V \in C^2(M)$, let $L_1 = \Delta + \nabla U$, $L_2 = \Delta + \nabla V$. Then $\alpha(L_1) \geqslant \exp[-\delta(U - V)] \, \alpha(L_2)$.

Lemma 2.3. Let $K \subset \mathbf{R}^d$ be a closed cube with side length r and each surface parallel to a coordinate plane. For $A \subset K$, let

$$H_A = \beta \sum_{F \subset A \cap \mathbf{Z}^d} J_F$$

$$L_A = \sum_{u \in A \cap \mathbf{Z}^d} (\Delta_u - \nabla_u H_A)$$

Let $\alpha(L_A) = \infty$ for $A \cap \mathbf{Z}^d = \varnothing$; then

$$\alpha(L_A) \geqslant \alpha_0(\beta) \exp[-\lambda\beta(r+1)^d/2], \qquad A \subset K$$

Proof. Suppose that $A \cap \mathbf{Z}^d \neq \varnothing$; let

$$\bar{L}_A = \sum_{u \in A \cap \mathbf{Z}^d} (\Delta_u - \beta \nabla_u J_{\{u\}})$$

By Lemma 2.1 we have $\alpha(\bar{L}_A) = \alpha_0(\beta)$. On the other hand,

$$\delta\left(H_A - \beta \sum_{u \in A \cap \mathbf{Z}^d} J_{\{u\}}\right) \leqslant \beta \sum_{F \subset K \cap \mathbf{Z}^d, |F| \geqslant 2} \delta(J_F)$$

$$\leqslant \frac{\beta}{2} \sum_{u \in K \cap \mathbf{Z}^d} \sum_{F \ni u, |F| \geqslant 2} \delta(J_F) \leqslant \frac{1}{2} \lambda\beta(r+1)^d$$

The proof is then completed by Lemma 2.2. ∎

Lemma 2.4. Suppose that $d \geqslant 2$ and L_A is defined in Lemma 2.3. For $m \in \mathbf{N}$, let $S_m = \sum_{i=0}^{m} d^i$. Then

$$\alpha(L_A) \geqslant \alpha_0(\beta) \exp[-2\lambda\beta R d c_d (r+1)^{d^{m+1}/S_m}], \qquad m \in \mathbf{N} \tag{2.1}$$

Proof. (a) Note that $d^{m+1}/S_m \in [d-1, d]$, $(a+b)^s \leqslant s(a^s + b^s)$ for $s \in [1, 2]$ and $a, b \geqslant 0$, and $(1+x^{-1})(x^{1/x} + x^{-1})$ is decreasing for $x > 0$; then

$$\begin{aligned}&[(d^{m+1}/S_m)^{S_m/S_{m+1}} + (d^{m+1}/S_m)^{-d^{m+1}/S_{m+1}}]^{S_{m+1}/d^{m+1}} \\ &\quad \leqslant (1 + S_m/d^{m+1})[(d^{m+1}/S_m)^{S_m/d^{m+1}} + S_m/d^{m+1}] \leqslant c_d\end{aligned}$$

(b) Suppose that $A \cap \mathbf{Z}^d \neq \varnothing$; for $s > 0$, let $\mathrm{i}(s)$ be the integer part of s. We equally divide K into $[\mathrm{i}(s)+1]^d$ many small cubes with side length $r/[\mathrm{i}(s)+1]$ and each surface parallel to a coordinate plane. Denote by K_i ($i \leqslant [\mathrm{i}(s)+1]^d$) all the small closed cubes and let $A_i = A \cap K_i$. Define

$$\begin{aligned}B(s) = \{u \in K\text{: there exist } i \neq j \text{ such that} \\ K_i \cap K_j \neq \varnothing \text{ and } d(u, K_i \cap K_j) < R\}\end{aligned}$$

Then $|B(s) \cap \mathbf{Z}^d| \leqslant 2Rd\mathrm{i}(s)(r+1)^{d-1}$. Let

$$V_s = \beta \sum_{F \subset A \cap \mathbf{Z}^d, F \cap B(s) \neq \varnothing, |F| \geqslant 2} J_F$$

We have

$$\delta(V_s) \leqslant \beta \sum_{u \in B(s) \cap \mathbf{Z}^d} \sum_{F \ni u, |F| \geqslant 2} \delta(J_F) \leqslant 2Rd\lambda\beta s(r+1)^{d-1} \tag{2.2}$$

Let

$$L_{A,s} = \sum_{u \in A \cap \mathbf{Z}^d} (\Delta_u - \nabla_u(H_A - V_s))$$

By Lemma 2.2 we have

$$\alpha(L_A) \geqslant \exp[-2Rd\lambda\beta s(r+1)^{d-1}]\, \alpha(L_{A,s}) \tag{2.3}$$

(c) For $i \leqslant [\mathrm{i}(s)+1]^d$, let $A_i^o = A_i \backslash B(s)$; then A_i^o is contained by a cube with side length $r/[\mathrm{i}(s)+1] - R$ and each surface parallel to a coordinate plane (note that $A_i^o = \varnothing$ if $r/[\mathrm{i}(s)+1] - R < 0$). Next, note that for each $F \subset A \cap \mathbf{Z}^d$ with $F \cap B(s) = \varnothing$ and $|F| \geqslant 2$, there holds $d(F) > R$ or $F \subset A_i^o$ for some i; then

$$H_A - V_s = \sum_i H_{A_i^o} + \beta \sum_{u \in B(s) \cap A} J_{\{u\}}$$

By Lemma 2.1 and (2.3) we obtain

$$\alpha(L_A) \geqslant \exp[-2Rd\lambda\beta s(r+1)^{d-1}] \min_i \{\alpha(L_{A_i^o}), \alpha_0(\beta)\} \tag{2.4}$$

By combining this with Lemma 2.3, we have

$$\begin{aligned}\alpha(L_A) &\geqslant \alpha_0(\beta) \exp\left[-d\lambda\beta\left(2Rs(r+1)^{d-1} + \frac{1}{2}\left(\frac{r}{\mathrm{i}(s)+1} - R + 1\right)^d\right)\right] \\ &\geqslant \alpha_0(\beta) \exp[-d\lambda\beta(2Rs(r+1)^{d-1} + (r+1)^d s^{-d}/2)]\end{aligned}$$

By taking $s = (d(r+1)/4R)^{1/(d+1)}$ and using (a), we obtain

$$\begin{aligned}\alpha(L_A) &\geqslant \alpha_0(\beta) \exp[-d\lambda\beta 2^{(d-1)/(d+1)} R^{d/d+1} \\ &\quad \times (d^{1/(d+1)} + d^{-d/(d+1)})(r+1)^{d^2/S_1}] \\ &\geqslant \alpha_0(\beta) \exp[-2\lambda\beta d R_d (r+1)^{d^2/S_1}]\end{aligned}$$

Then (2.1) holds for $m = 1$.

(d) Suppose that (2.1) holds for some $m \in \mathbf{N}$; we need only to prove it for $m+1$. Actually, let A_i^o be defined in (b); since A_i^o is contained by a cube with side length $r/[\mathrm{i}(s)+1] - R$, the assumption implies

$$\begin{aligned}\alpha(L_{A_i^o}) &\geqslant \alpha_0(\beta) \exp\left[-2Rd\lambda\beta c_d \left(\frac{r}{\mathrm{i}(s)+1} - R + 1\right)^{d^{m+1}/S_m}\right] \\ &\geqslant \alpha_0(\beta) \exp[-2Rd\lambda\beta c_d (r+1)^{d^{m+1}/S_m} s^{-d^{m+1}/S_m}]\end{aligned}$$

By (2.4) and taking

$$s = [c_d d^{m+1}/S_m]^{S_m/S_{m+1}} (r+1)^{1/S_{m+1}}$$

and using (a), we get

$$\begin{aligned}\alpha(L_A) &\geqslant \alpha_0(\beta) \exp[-2Rd\lambda\beta(s(r+1)^{d-1} + c_d s^{-d^{m+1}/S_m}(r+1)^{d^{m+1}/S_m})] \\ &\geqslant \alpha_0(\beta) \exp[-2dR\lambda\beta c_d (r+1)^{d^{m+2}/S_{m+1}}]\end{aligned}$$

Therefore (2.1) holds for $m+1$. ∎

Proof of Theorem 1.1. The estimate for the free boundary case follows from Lemma 2.4 by letting $m \to \infty$. To estimate $\alpha(L_{\Lambda|y})$, let $K \subset \mathbf{R}^d$ be the cube with side length l and $K \cap \mathbf{Z}^d = \Lambda$. Then

$$\delta(H_{\Lambda|y} - H_K) \leqslant \beta \sum_{F \cap K \neq \varnothing, F \cap K^c \neq \varnothing} \delta(J_F)$$

$$\leqslant \beta \sum_{u \in \Lambda, d(u, \partial K) < R} \lambda \leqslant 2dR\lambda\beta(l+1)^{d-1}$$

By Lemmas 2.2 and 2.4 we have

$$\alpha(L_{\Lambda|y}) \geqslant \exp[-2dR\lambda\beta(l+1)^{d-1}]\, \alpha(L_K)$$

$$\geqslant \alpha_0(\beta) \exp[-2dR\lambda\beta(l+1)^{d-1}]$$

$$\times \exp[-2\lambda\beta R d c_d (l+1)^{d^{m+1}/S_m}]$$

Then the proof is completed by letting $m \to \infty$. ∎

3. PROOF OF THEOREM 1.2

The basic idea of the proof is to construct a set of configurations for which a smoothed indicator function has small energy. Thus, the key point is to construct this set and to estimate its measure. To do this, we first construct the set of configurations by relating the present configurations to discrete contour ones, then give estimates on the measure of this set by using FKG and GKS inequalities; finally, the rest of the proof follows from the method of ref. 13.

For $\Theta \subset \Lambda$, let Q be the union of all the closed unit cubes in $\mathbf{R}^d$ with center points in Θ and each face parallel to a coordinate plane. Denote by $\partial\Theta = \partial Q$ the boundary of Q. Recall that a Peierls contour is the boundary of a simply connected $\Theta \subset \Lambda$. Let Γ be the set of all Peierls contours. For $\gamma \in \Gamma$, let $\Theta(\gamma)$ denote the simply connected $\Theta \subset \Lambda$ with $\partial\Theta = \gamma$ and let $Q(r)$ be the corresponding Q, and define

$$\partial^+\gamma = \{u \in \Theta(\gamma) \colon d(u, \gamma) = 1/2\}$$

$$\partial^-\gamma = \{u \in \Lambda \backslash \Theta(\gamma) \colon d(u, \gamma) = 1/2\}$$

$$\bar{\gamma} = \partial^+\gamma \cup \partial^-\gamma$$

Finally, for $u \in \Lambda$, let $\Gamma(u) = \{\gamma \in \Gamma \colon u \in \Theta(\gamma)\}$. The following result is an improvement of ref. 13, Lemma 2.1.

Proposition 3.1. For $\varepsilon \in (0, 1)$, let

$$\kappa(\varepsilon) = \varepsilon^{1/2(d-1)} \vee [1 - \varepsilon^{1/2(d-1)}(1 - \varepsilon^{1/2(d-1)})/2d]$$

If $\gamma \in \Gamma$ satisfying $|\gamma| < \varepsilon\, |\partial\Lambda| = 2d\varepsilon(l+1)^{d-1}$, then $|\gamma \cap \partial\Lambda| < \kappa(\varepsilon)\, |\gamma|$.

To prove Proposition 3.1, we need the following isoperimetric inequality.

Lemma 3.1. For $\Theta \subset \Lambda$, we have $|\partial\Theta| \geqslant 2d\, |\Theta|^{(d-1)/d}$.

Proof. Simply refer to ref. 10, (3). ∎

Proof of Proposition 3.1. (a) Let $\sigma = \varepsilon^{1/2(d-1)}$; if $|\gamma \cap \partial\Lambda| \geqslant \sigma\, |\gamma|$, there exists a face F of $\partial\Lambda$ such that $|F \cap \gamma| \geqslant \sigma\, |\gamma|/2d$. Without loss of generality, we assume that $\Lambda = [a, a+l]^d \cap \mathbf{Z}^d$ and $F = \{x_1 = a + l + 1/2\} \cap \partial\Lambda$. For $r \in [a - 1/2, a + l + 1/2]$, let $Q_r = \{x_1 = r\} \cap Q(\gamma)$. Suppose that $|Q_r| \geqslant \sigma^2\, |\gamma|/2d$ for all $r \in [a - 1/2, a + l + 1/2]$; then

$$|Q(\gamma)| = \int_{a-1/2}^{a+l+1/2} |Q_r|\, dr \geqslant \sigma^2 (l+1)\, |\gamma|/2d$$

By Lemma 3.1 we have

$$\begin{aligned} |\gamma| = |\partial Q(\gamma)| &\geqslant 2d(\sigma^2(l+1)\, |\gamma|/2d)^{(d-1)/d} \\ &= \sigma^{2(d-1)/d}\, |\partial\Lambda|^{1/d}\, |\gamma|^{(d-1)/d} \end{aligned}$$

This implies $|\gamma| \geqslant \sigma^{2(d-1)}\, |\partial\Lambda| = \varepsilon\, |\partial\Lambda|$, which contradicts the assumption of Proposition 3.1. Therefore, there must exist $r_0 \in [a - 1/2, a + l + 1/2)$ such that $|Q_{r_0}| < \sigma^2\, |\gamma|/2d$.

(b) Let γ' be the boundary of $Q(\gamma) \cap \{x_1 \geqslant r_0\}$. Note that the projection of $Q_{a+l+1/2}$ to $\{x_1 = r_0\}$ is contained by that of $\gamma' \backslash \partial\Lambda$ to the same plane; then

$$\begin{aligned} |\gamma \backslash \partial\Lambda| \geqslant |\gamma' \backslash \partial\Lambda| - |Q_{r_0}| &> |Q_{a+l+1/2}| - \sigma^2\, |\gamma|/2d \\ &\geqslant \sigma(1 - \sigma)\, |\gamma|/2d \end{aligned}$$

Therefore

$$|\gamma \cap \partial\Lambda| < (1 - \sigma(1 - \sigma)/2d)\, |\gamma| \quad ∎$$

Given $\Theta \subset \Lambda$, let

$$\mathcal{M}(\Theta) = \{f \in C([0, 1]^\Theta): f(x) \geqslant f(y) \text{ if } x_u \geqslant y_u \text{ for all } u \in \Theta\}$$

Next, for $y \in [-1, 1]^{\Theta^c}$, define $\pi_{\Theta, y} \in \mathscr{P}([-1, 1]^{\Theta})$ as follows:

$$\pi_{\Theta, y}(dx_\Theta) = Z_{\Theta^c}(y)^{-1} \exp[-H_\Lambda(x_\Theta \times y)]\, dx_\Theta$$

$$Z_{\Theta^c}(y) = \int_{[-1,1]^\Theta} \exp[-H_\Lambda(x_\Theta \times y)]\, dx_\Theta, \qquad dx_\Theta = \prod_{u \in \Theta} dx_u$$

Then the GKS inequality implies[5,6]

$$\pi_{\Theta, y}(x_u) \geqslant 0, \qquad u \in \Theta, \quad y \in [0, 1]^{\Theta^c} \tag{3.1}$$

Lemma 3.2. For $\Theta \subset \Lambda$, we have $Z_\Theta \in \mathscr{M}(\Theta)$.

Proof. For $u \in \Theta$, it follows from (3.1) that

$$\frac{\partial Z_\Theta(y)}{\partial y_u} = \beta \sum_{v \in \Theta^c:\, |u - v| = 1} Z_\Theta(y)\, \pi_{\Theta^c, y}(x_v) + \beta \sum_{v \in \Theta:\, |u - v| = 1} y_v Z_\Theta(y) \geqslant 0 \quad \blacksquare$$

Lemma 3.3. For $\Theta \subset \Lambda$, $u \in \Theta$, and $r \in [0, 1)$, we have

$$\int_{-r}^{r} Z_\Theta(x_\Theta)\, dx_u \leqslant r \exp(2d\beta r)\, Z_{\Theta \setminus u}(x_{\Theta \setminus u})$$

Proof. For $x_\Theta \in [-1, 1]^\Theta$ with $x_u \in [0, r]$,

$$Z_\Theta(x_\Theta) \leqslant \exp(2d\beta r) \int_{[-1,1]^{\Theta^c}} \exp[-H_{\{u\}^c}(x_{\{u\}^c})]\, dx_{\Theta^c} \tag{3.2}$$

Next,

$$\begin{aligned} Z_{\Theta \setminus u}(x_{\Theta \setminus u}) &= \int_{[-1,1]^{\Theta^c \cup \{u\}}} \exp[-H_\Lambda(x)]\, dx_{\Theta^c \cup \{u\}} \\ &= \int_{[-1,1]^{\Theta^c \cup \{u\}}} \exp\left(\beta \sum_{v:\, |u - v| = 1} x_u x_v\right) \\ &\quad \times \exp[-H_{\{u\}^c}(x_{\{u\}^c})]\, dx_{\Theta^c \cup \{u\}} \\ &= \int_{[-1,1]^{\Theta^c \cup \{u\}}} \exp\left(-\beta \sum_{v:\, |u - v| = 1} x_u x_v\right) \\ &\quad \times \exp[-H_{\{u\}^c}(x_{\{u\}^c})]\, dx_{\Theta^c \cup \{u\}} \end{aligned}$$

In the last step we used the integral transformation $x_u = -x_u$. Note that $e^r + e^{-r} \geqslant 2$; we have

$$Z_{\Theta\setminus u}(x_{\Theta\setminus u}) \geqslant 2 \int_{[-1,1]^{\Theta^c}} \exp[-H_{\{u\}^c}(x_{\{u\}^c})]\, dx_{\Theta^c}$$

The proof is then completed by (3.2). ∎

Now, we try to connect a spin configuration $x \in [-1, 1]^\Lambda$ with Peierls contours. For $x \in [-1, 1]^\Lambda$ and $r \in [0, 1)$, we mark + or − at u with respect to $x_u > r$ or $x_u \leqslant r$. The set of all Peierls contours determined by x is $\Gamma_x^r := \partial\{u: x_u > r\}$. In contrast to the case of spin space $\{-1, 1\}^{\mathbf{Z}^d}$, here Γ_x^r can be equal to Γ_y^r for $x \neq y$.

For $r \in [0, 1)$, let

$$S_r = \{x \in [-1, 1]^\Lambda\text{: there exists } \gamma \in \Gamma_x^r \text{ such that } |\gamma| \geqslant 0.8\, |\partial\Lambda|\}$$

The following lemma is similar to ref. 13, Lemma 2.2.

Lemma 3.4. Let $\delta = (\beta\, |\partial\Lambda|)^{-1}$; we have $\mu_\Lambda(S_\delta) \geqslant 0.4$ for β sufficiently large independent of Λ.

Proof. (a) Fix $u_0 \in \Lambda$; we have $\mu_\Lambda(x_{u_0} > 0) = \mu_\Lambda(x_{u_0} \leqslant 0) = 1/2$. By this and Lemma 3.3, $\mu_\Lambda(x_{u_0} > \delta) \geqslant 1/2 - e\delta$. Then

$$\begin{aligned}\mu_\Lambda(S_\delta) &\geqslant \tfrac{1}{2} - e\delta - \mu_\Lambda(x_{u_0} > \delta \text{ and } |\gamma| < 0.8\, |\partial\Lambda| \\ &\qquad \text{for all} \quad \gamma \in \Gamma_x^\delta \cap \Gamma(u_0)) \\ &\geqslant \tfrac{1}{2} - e\delta - \sum_{0.8\, |\partial\Lambda| > h \geqslant 2d} \; \sum_{\gamma \in \Gamma(u_0):\, |\gamma| = h} \mu_\Lambda(\gamma \in \Gamma_x^\delta) \end{aligned} \tag{3.3}$$

For given γ with $|\gamma| = h < 0.8\, |\partial\Lambda|$, we have $|\gamma\setminus\gamma \cap \partial\Lambda| > (1 - \kappa(0.8))\, h$. Let

$$A = \{(u, v) \in \partial^+\gamma \times \partial^-\gamma\text{: } |u - v| = 1\}$$

Then $|A| > (1 - \kappa(0.8))\, h$ and

$$\begin{aligned}\mu_\Lambda(\gamma \in \Gamma_x^\delta) &\leqslant Z_\Lambda^{-1} \int_{[\delta,1]^{\partial^+\gamma} \times [-1,\delta]^{\partial^-\gamma} \times [-1,1]^{\bar\gamma^c}} \exp[-H_\Lambda(x)]\, dx \\ &\leqslant Z_\Lambda^{-1} \prod_{(u,v) \in A} \left(\int_0^1 \int_{-1}^0 + \int_0^1 \int_0^\delta\right) \\ &\qquad \times \int_{[-1,1]^{\bar\gamma^c}} \exp[-H_\Lambda(x)]\, dx \end{aligned} \tag{3.4}$$

(b) For $A' \subset A$ with $|A'| = s$, let $G = \{u, v: (u, v) \in A'\}$. By first using the transformation $x_{\Theta(\gamma)^c} = -x_{\Theta(\gamma)^c}$ and then Lemma 3.3, we have

$$I := \prod_{(u,v)\in A'} \left(\int_0^1 \int_{-1}^0\right) \prod_{(u,v)\in A\backslash A'} \left(\int_0^1 \int_0^\delta\right)$$

$$\times \int_{[-1,1]^{\gamma^c}} \exp[-H_A(x)]\, dx$$

$$\leqslant \exp(4d)\, \delta^{|A|-s} \int_{[0,1]^G} \exp\left(-2\beta \sum_{(u,v)\in A'} x_u x_v\right) Z_G(x_G)\, dx_G$$

Since Z_G is increasing on $[0, 1]^G$ and dx_G has positive correlations,[5,6] we have

$$I \leqslant Z_A \exp(4d)\, \delta^{|A|-s} \int_{[0,1]^G} \exp\left(-2\beta \sum_{(u,v)\in A'} x_u x_v\right) dx_G$$

Note that for each $(u, v) \in A'$,

$$|\{(u', v') \in A': \{u, v\} \cap \{u', v'\} \neq \varnothing\}| \leqslant 4d - 1$$

We may choose $B \subset A'$ such that $|B| \geqslant s/4d$ and for $(u, v) \neq (u', v') \in B$, $\{u, v\} \cap \{u', v'\} = \varnothing$. Therefore

$$I \leqslant Z_A \exp(4d)\, \delta^{|A|-s} \left(\int_0^1 \int_0^1 \exp(-2\beta xy)\, dx\, dy\right)^{s/4d} \tag{3.5}$$

On the other hand, for $\beta \geqslant 1/2$ we have

$$\int_0^1 \int_0^1 \exp(-2\beta xy)\, dx\, dy$$

$$= \int_0^1 \frac{1}{2\beta x} [1 - \exp(-2\beta x)]\, dx$$

$$< (2\beta)^{-1} + \int_{(2\beta)^{-1}}^1 (2\beta x)^{-1}\, dx$$

$$= (2\beta)^{-1} [1 + \log(2\beta)]$$

By (3.4) and (3.5) we obtain

$$\mu_\Lambda(\gamma\in\Gamma_x^\delta)\leqslant \exp(4d)\sum_{s=0}^{|\Lambda|}\binom{|\Lambda|}{s}\delta^{|\Lambda|-s}\{[1+\log(2\beta)](2\beta)^{-1}\}^{s/4d}$$
$$\leqslant \exp(4d)(\beta^{-1}\log\beta)^{\alpha_1 h} \tag{3.6}$$

for some constant $\alpha_1>0$ and large β.

(c) Note that the number of Peierls contours with $|\gamma|=h$ and $u_0\in\Theta(\gamma)$ is no more than $\frac{1}{2}h(3(2d-3))^h$ (see ref. 13, Proof of Lemma 2.2); by (3.3) and (3.6) we obtain

$$\mu_\Lambda(S_\delta)\geqslant\frac{1}{2}-e\delta-\frac{\exp(4d)}{2}\sum_{h\geqslant 2d}h(3(2d-3))^h(\beta^{-1}\log\beta)^{\alpha_1 h}\geqslant 0.4$$

for sufficiently large β. ▮

Lemma 3.5. If $l/(l+1)>0.6^{1/(d-1)}$, then $\mu_\Lambda(S_0^c)\geqslant 0.4$ for large β.

Proof. Let

$$S'=\{x:|\{u\in\partial^+\partial\Lambda: x_u\leqslant 0\}|\geqslant|\partial^+\partial\Lambda|/2\}$$
$$S''=\{x:|\{u\in\partial^+\partial\Lambda: x_u\geqslant 0\}|\geqslant|\partial^+\partial\Lambda|/2\}$$

Then $S'=-S''$, $S'\cup S''=[-1,1]^\Lambda$, and so $\mu_\Lambda(S')=\mu_\Lambda(S'')\geqslant 1/2$. For $x\in S_0\cap S'$, there exists $\gamma\in\Gamma_x^0$ such that $|\gamma|\geqslant 0.8\,|\partial\Lambda|$. Note that $x\in S'$ and

$$|\partial^+\partial\Lambda|\geqslant 2dl^{d-1}=\left(\frac{l}{l+1}\right)^{d-1}|\partial\Lambda|>0.6\,|\partial\Lambda|$$

We have

$$|\gamma\cap\partial\Lambda|\leqslant|\partial\Lambda|-\frac{|\partial^+\partial\Lambda|}{2}<0.7\,|\partial\Lambda|\leqslant\frac{7}{8}\,|\gamma|$$

Then the proof of Lemma 3.4 gives

$$\mu_\Lambda(S_0\cap S')\leqslant\sum_{u\in\Lambda}\sum_{h\geqslant 0.8\,|\partial\Lambda|}\sum_{\gamma\in\Gamma(u):\,|\gamma|=h,\,|\gamma\cap\partial\Lambda|<7h/8}\mu_\Lambda(\gamma\in\Gamma_x^0)$$
$$\leqslant\frac{(l+1)^d}{2}\sum_{h\geqslant 0.8\,|\partial\Lambda|}h(3(2d-3))^h(\beta^{-1}\log\beta)^{-\alpha_2 h}\leqslant 0.1$$

for some constant $\alpha_2>0$ and large β. Hence $\mu_\Lambda(S_0^c)\geqslant 0.5-\mu_\Lambda(S_0\cap S')\geqslant 0.4$ for sufficiently large β. ▮

Lemma 3.6. Let

$$\alpha_3 = [3-\kappa(0.8)]/2[2-\kappa(0.8)], \qquad \alpha_4 = [1-\kappa(0.8)]/16d.$$

Suppose that $|\gamma| \geqslant \alpha_3 |\partial\Lambda|$, $|\gamma \cap \partial\Lambda| \geqslant \alpha_3 |\gamma|$, and the length of each contour contained in $Q(\gamma)^o$ is $\leqslant \alpha_4 |\gamma|$. To change γ into contours with length less than $0.8 |\partial\Lambda|$ without changing spins "−" and spins in $\Theta(\gamma)^c$, one must replace more than three "+" by "−" in $\Theta(\gamma)$ for large l.

Proof. Suppose that we have replaced just m many "+" by "−" in $\Theta(\gamma)$ and γ has been changed into some new contours $\gamma_1,..., \gamma_s$ with $|\gamma_i| < 0.8 |\partial\Lambda|$, $i \leqslant s$; then

$$\sum_{i=1}^{s} |\gamma_i \cap \gamma| \leqslant \sum_{i=1}^{s} |\gamma_i \cap \partial\Lambda| + (1-\alpha_3)|\gamma|$$
$$< \kappa(0.8) \sum_{i=1}^{s} |\gamma_i| + (1-\alpha_3)|\gamma| \tag{3.7}$$

Next, let $n = |\{u \in \partial^+\gamma$: the spin at u was changed$\}|$; by (3.7) we have

$$|\gamma| \leqslant \sum_{i=1}^{s} |\gamma \cap \gamma_i| + n(2d-1) < (1-\alpha_3)|\gamma| + n(2d-1) + \kappa(0.8) \sum_{i=1}^{s} |\gamma_i|$$

This implies

$$\sum_{i=1}^{s} |\gamma_i| \geqslant \alpha_3 |\gamma| - n(2d-1) \tag{3.8}$$

Let Γ_1 be the union of contours contained in $Q(\gamma)^o$ and let $\Gamma_2 = (\bigcup \gamma_i) \backslash (\Gamma_1 \cup \gamma)$. Note that there are at most $2d$ many γ_i which are connected with γ by changing one spin. Hence $|(\bigcup \gamma_i) \cap \Gamma_1| \leqslant 2dm\alpha_4 |\gamma|$. Combining this with (3.7) and (3.8), we have

$$|\Gamma_2| \geqslant \sum_{i=1}^{s} |\gamma_i \backslash \gamma| - 2dm\alpha_4 |\gamma|$$
$$= \sum_{i=1}^{s} (|\gamma_i| - |\gamma_i \cap \gamma|) - 2dm\alpha_4 |\gamma|$$
$$> [1-\kappa(0.8)] \sum_{i=1}^{s} |\gamma_i| - (1-\alpha_3+2dm\alpha_4)|\gamma|$$
$$\geqslant \{\alpha_3[2-\kappa(0.8)] - 1 - 2dm\alpha_4\} |\gamma| - [1-\kappa(0.8)](2d-1)\, n$$

Note that $m \geqslant [1/(2d)] |\Gamma_2|$ and $m \geqslant n$; we have

$$m \geqslant \frac{\alpha_3[2-\kappa(0.8)]-1}{4d+2d\alpha_4 |\gamma|} |\gamma| = \frac{4[1-\kappa(0.8)] |\gamma|}{32d+[1-\kappa(0.8)] |\gamma|} > 3$$

for large l. ∎

Proof of Theorem 1.2. We need only to prove the result for large l and large β.

(a) Since $S_0^c \subset \mathbf{R}^A$ is a bounded domain, for $r \in (0, \delta]$ we choose $f \in C^\infty(\mathbf{R}^A)$ such that $\|\nabla_A f\| \leqslant 2/r$ and

$$1 \geqslant f(x) \begin{cases} =1 & \text{if } \ x \in S_0^c \\ =0 & \text{if } \ d(x, S_0^c) \geqslant r \\ \geqslant 0 & \text{otherwise} \end{cases}$$

Note that $d(S_r, S_0^c) \geqslant r$; then $f|_{S_r} \equiv 0$. Therefore

$$\text{gap}(L_A) \leqslant \frac{\mu_A(\|\nabla_A f\|^2)}{\mu_A(f^2)-\mu_A(f)^2} \leqslant \frac{4\mu_A(S_r^c \cap S_0)}{r^2[\mu_A(S_0^c)-\mu_A(S_r^c)^2]}$$

By Lemmas 3.4 and 3.5, we have

$$\mu_A(S_0^c)-\mu_A(S_r^c)^2 \geqslant 0.4-0.6^2=0.04$$

for sufficiently large β. Hence there exists $c_1(\beta)>0$ such that

$$\text{gap}(L_A) \leqslant c_1(\beta)\, r^{-2} \mu_A(S_r^c \cap S_0) \tag{3.9}$$

for large β.

(b) For $x \in [-1, 1]^A$, we call $\gamma \in \Gamma_x^0$ a good contour if

$$|\gamma| \geqslant 0.8\, |\partial A|, \qquad |\gamma \cap \partial A| > \kappa(\alpha_3)\, |\gamma|$$

and

$$\max\{|\gamma_i|: \gamma_i \in \Gamma_x^0, \gamma_i \subset Q(\gamma)^o\} \leqslant \alpha_4\, |\gamma|$$

Let $\alpha_5 = 0.8 \min\{\alpha_4, 1-\kappa(\alpha_3)\}$ and set

$$T_1 = \{x \in S_0\text{: there exists a good contour in } \Gamma_x^0\}$$

$$T_2 = \{x\text{: there exists } \gamma \in \Gamma_x^0 \text{ such that } |\gamma \backslash \partial A| \geqslant \alpha_5\, |\partial A|\}$$

For $x \in T_1^c \cap S_0$, there exists $\gamma \in \Gamma_x^0$ such that $|\gamma| \geqslant 0.8\, |\partial A|$ and either $|\gamma \cap \partial A| \leqslant \kappa(\alpha_3)\, |\gamma|$ or there exists $\gamma' \in \Gamma_x^0$ and $\gamma' \subset Q(\gamma)^o$ satisfying

$|\gamma'| \geqslant \alpha_4 |\gamma|$. Hence, there exists $\gamma'' \in \Gamma_x^0$ such that $|\gamma'' \backslash \partial \Lambda| \geqslant \alpha_5 |\partial \Lambda|$. Therefore $T_1^c \cap S_0 \subset T_2$. Thus

$$\mu_\Lambda(S_r^c \cap S_0) \leqslant \mu_\Lambda(S_0 \cap T_1^c) + \mu_\Lambda(S_r^c \cap T_1) \leqslant \mu_\Lambda(T_2) + \mu_\Lambda(S_r^c \cap T_1) \quad (3.10)$$

(c) Obviously, the proof of Lemma 3.4 gives (replacing δ by 0)

$$\begin{aligned}\mu_\Lambda(T_2) &\leqslant \sum_{\gamma:\, |\gamma \backslash \partial \Lambda| \geqslant \alpha_5 |\partial \Lambda|} \mu_\Lambda(\gamma \in \Gamma_x^0) \\ &\leqslant \tfrac{1}{2}(l+1)^d \sum_{h \geqslant \alpha_5 |\partial \Lambda|} h(3(2d-1))^h (\beta^{-1} \log \beta)^{\alpha_6 h} \\ &\leqslant c_2(\beta) \exp[-\alpha_7 \log(\beta/\log \beta)(l+1)^{d-1}] \end{aligned} \quad (3.11)$$

for some $c_2(\beta)$, α_6, and $\alpha_7 > 0$.

Next, for $x \in T_1 \cap S_r^c$, there exists a good contour $\gamma \in \Gamma_x^0$ and the length of each contour in Γ_x^r is $< 0.8 |\partial \Lambda|$. Note that $\kappa(\alpha_3) \geqslant \alpha_3$ and by Proposition 3.1, $|\gamma| \geqslant \alpha_3 |\partial \Lambda|$; Lemma 3.6 implies for large l that

$$T_1 \cap S_r^c \subset \{\text{there exist three different sites } u_1, u_2, u_3 \text{ such that } x_{u_i} \in (0, r], i \leqslant 3\}$$

By Lemma 3.3 we have

$$\mu_\Lambda(T_1 \cap S_r^c) \leqslant \binom{|\Lambda|}{3} r^3 \exp(6d\beta r) \leqslant \exp(6d)(l+1)^{3d} r^3 \quad (3.12)$$

Take $r = \exp[-\frac{1}{4}\alpha_7 \log(\beta/\log \beta)(l+1)^{d-1}]$; then $r < \delta$ for large l. By (3.9)–(3.12), we obtain

$$\begin{aligned}\operatorname{gap}(L) &\leqslant c_1(\beta) \exp(6d)(l+1)^{3d} \\ &\quad \times \exp[-\tfrac{1}{4}\alpha_7 \log(\beta/\log \beta)(l+1)^{d-1}] \\ &\quad + c_1(\beta)\, c_2(\beta) \exp[-\tfrac{1}{4}\alpha_7 \log(\beta/\log \beta)(l+1)^{d-1}] \\ &\leqslant c(\beta) \exp[-\alpha \log(\beta/\log \beta)(l+1)^{d-1}]\end{aligned}$$

for some $c(\beta) > 0$ and $\alpha > 0$. ∎

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